B-math 2nd year Mid Term
Subject : Analysis III

Time : 3.00 hours Max.Marks 50.

1. Let $Q = [0, 1] \times [0, 1]$ and define $f(x, y)$ on Q as follows :

$$f(x,y) = \begin{cases} x+y & \text{if } (x,y) \in Q, x+y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on Q and evaluate the double integral $\iint\limits_{Q} f$ of f over Q.

(10)

2. Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let C be a piecewise smooth Jordan curve lying in S taken in the clockwise direction. Evaluate $\int\limits_{C} Pdx + Qdy$, where $P(x, y) := \frac{y}{x^2+y^2}$ and $Q(x, y) := \frac{-x}{x^2+y^2}$. Hint : consider (separately) the case $(0, 0)$ is inside (respy. outside) C.

(10)

3. Let $f : [0, \infty) \to [0, \infty)$ be continuous and define $g(x, y) := f(x^2 + y^2)$ $(x, y) \in \mathbb{R}^2$. Let $R := [-r, r] \times [-r, r]$. Show that

$$\lim_{r \uparrow \infty} \iint\limits_{R} g(x,y)dxdy = \pi \int_0^\infty f(u)du$$

where the (improper) integral in the RHS is evaluated as a limit of integrals over finite intervals. (10)

4. Let R be a region in $\mathbb{R}^2$ and C, a piecewise smooth Jordan curve, be its boundary. Let $\alpha(t) := (X(t), Y(t)), a \leq t \leq b$ be a parametrization of C. Show that

$$area(R) = \frac{1}{2} \int_a^b \{X(t)Y'(t) - Y(t)X'(t)\}dt$$

where the prime denotes differentiation. (10)

5. If ϕ and ψ are both potential functions for a continuous vector field f on an open connected set S in $\mathbb{R}^n$ show that $\phi - \psi$ is a constant. Hint : Show that there exists $C \in \mathbb{R}$ such that the set $S_1 = \{(x, y) \in S : \phi(x, y) = \psi(x, y) + C\}$ is a non empty open set in S. (10)